4.3 Rotations

Essential Question

How can you rotate a figure in a coordinate plane?

EXPLORATION 1 Rotating a Triangle in a Coordinate Plane

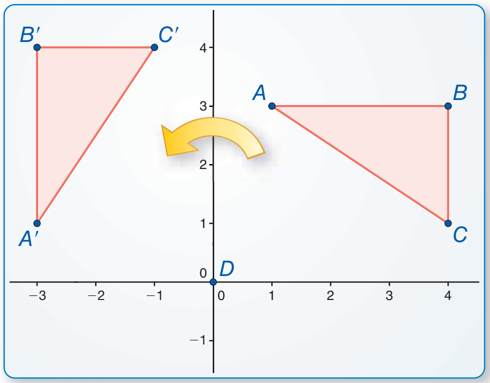
Work with a partner.

a. Use dynamic geometry software to draw any triangle and label it $\triangle ABC$.

b. *Rotate* the triangle 90° counterclockwise about the origin to form $\triangle A'B'C'$.

c. What is the relationship between the coordinates of the vertices of $\triangle ABC$ and those of $\triangle A'B'C'$?

d. What do you observe about the side lengths and angle measures of the two triangles?



Sample

Points
$A(1, 3)$
$B(4, 3)$
$C(4, 1)$
$D(0, 0)$
Segments
$AB = 3$
$BC = 2$
$AC = 3.61$
Angles
$m\angle A = 33.69°$
$m\angle B = 90°$
$m\angle C = 56.31°$

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to use previously established results in constructing arguments.

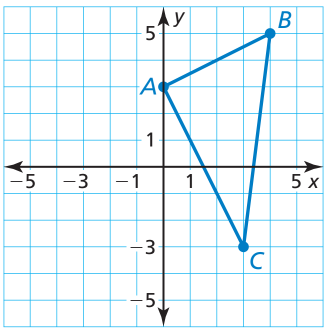
EXPLORATION 2 Rotating a Triangle in a Coordinate Plane

Work with a partner.

a. The point (x, y) is rotated 90° counterclockwise about the origin. Write a rule to determine the coordinates of the image of (x, y).

b. Use the rule you wrote in part (a) to rotate $\triangle ABC$ 90° counterclockwise about the origin. What are the coordinates of the vertices of the image, $\triangle A'B'C'$?

c. Draw $\triangle A'B'C'$. Are its side lengths the same as those of $\triangle ABC$? Justify your answer.



EXPLORATION 3 Rotating a Triangle in a Coordinate Plane

Work with a partner.

a. The point (x, y) is rotated 180° counterclockwise about the origin. Write a rule to determine the coordinates of the image of (x, y). Explain how you found the rule.

b. Use the rule you wrote in part (a) to rotate $\triangle ABC$ (from Exploration 2) 180° counterclockwise about the origin. What are the coordinates of the vertices of the image, $\triangle A'B'C'$?

Communicate Your Answer

4. How can you rotate a figure in a coordinate plane?

5. In Exploration 3, rotate $\triangle A'B'C'$ 180° counterclockwise about the origin. What are the coordinates of the vertices of the image, $\triangle A''B''C''$? How are these coordinates related to the coordinates of the vertices of the original triangle, $\triangle ABC$?

4.3 Lesson

Core Vocabulary

rotation, *p. 190*
center of rotation, *p. 190*
angle of rotation, *p. 190*
rotational symmetry, *p. 193*
center of symmetry, *p. 193*

What You Will Learn

- Perform rotations.
- Perform compositions with rotations.
- Identify rotational symmetry.

Performing Rotations

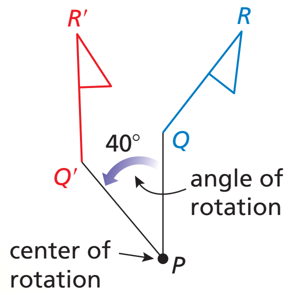
Core Concept

Rotations

A **rotation** is a transformation in which a figure is turned about a fixed point called the **center of rotation**. Rays drawn from the center of rotation to a point and its image form the **angle of rotation**.

A rotation about a point P through an angle of $x°$ maps every point Q in the plane to a point Q' so that one of the following properties is true.

- If Q is not the center of rotation P, then $QP = Q'P$ and $m\angle QPQ' = x°$, or
- If Q is the center of rotation P, then $Q = Q'$.



Direction of rotation

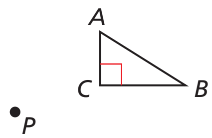
clockwise

counterclockwise

The figure above shows a 40° counterclockwise rotation. Rotations can be *clockwise* or *counterclockwise*. In this chapter, all rotations are counterclockwise unless otherwise noted.

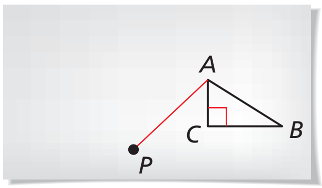
EXAMPLE 1 Drawing a Rotation

Draw a 120° rotation of $\triangle ABC$ about point P.

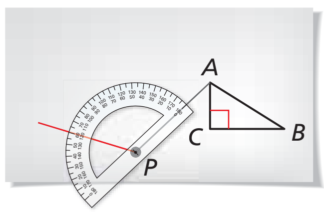


SOLUTION

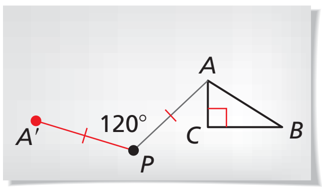
Step 1 Draw a segment from P to A.



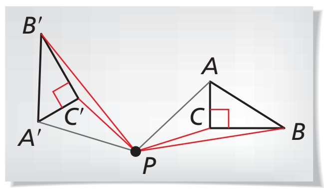
Step 2 Draw a ray to form a 120° angle with $\overline{PA}$.



Step 3 Draw A' so that $PA' = PA$.



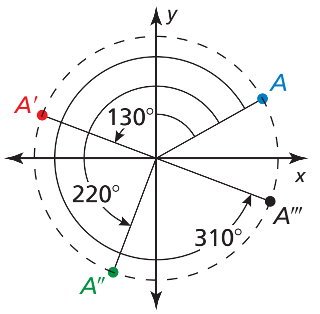
Step 4 Repeat Steps 1–3 for each vertex. Draw $\triangle A'B'C'$.



USING ROTATIONS

You can rotate a figure more than 360°. The effect, however, is the same as rotating the figure by the angle minus 360°.

You can rotate a figure more than 180°. The diagram shows rotations of point A 130°, 220°, and 310° about the origin. Notice that point A and its images all lie on the same circle. A rotation of 360° maps a figure onto itself.



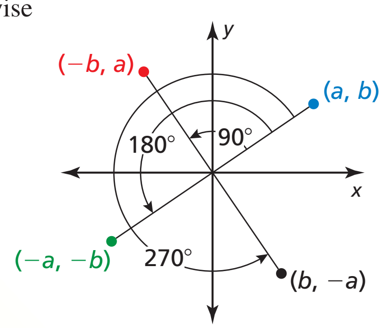
You can use coordinate rules to find the coordinates of a point after a rotation of 90°, 180°, or 270° about the origin.

Core Concept

Coordinate Rules for Rotations about the Origin

When a point (a, b) is rotated counterclockwise about the origin, the following are true.

- For a rotation of 90°, $(a, b) \rightarrow (-b, a)$.
- For a rotation of 180°, $(a, b) \rightarrow (-a, -b)$.
- For a rotation of 270°, $(a, b) \rightarrow (b, -a)$.



EXAMPLE 2 Rotating a Figure in the Coordinate Plane

Graph quadrilateral $RSTU$ with vertices $R(3, 1)$, $S(5, 1)$, $T(5, -3)$, and $U(2, -1)$ and its image after a 270° rotation about the origin.

SOLUTION

Use the coordinate rule for a 270° rotation to find the coordinates of the vertices of the image. Then graph quadrilateral $RSTU$ and its image.

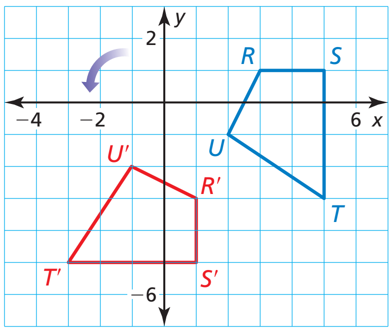
$$(a, b) \rightarrow (b, -a)$$

$$R(3, 1) \rightarrow R'(1, -3)$$

$$S(5, 1) \rightarrow S'(1, -5)$$

$$T(5, -3) \rightarrow T'(-3, -5)$$

$$U(2, -1) \rightarrow U'(-1, -2)$$



Monitoring Progress

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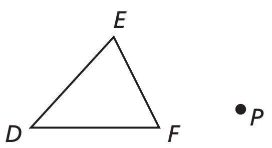
1. Trace $\triangle DEF$ and point P. Then draw a 50° rotation of $\triangle DEF$ about point P.



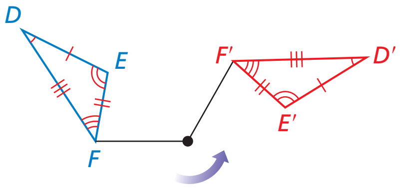
2. Graph $\triangle JKL$ with vertices $J(3, 0)$, $K(4, 3)$, and $L(6, 0)$ and its image after a 90° rotation about the origin.

Performing Compositions with Rotations

Postulate

Postulate 4.3 Rotation Postulate

A rotation is a rigid motion.



Because a rotation is a rigid motion, and a rigid motion preserves length and angle measure, the following statements are true for the rotation shown.

- $DE = D'E'$, $EF = E'F'$, $FD = F'D'$
- $m\angle D = m\angle D'$, $m\angle E = m\angle E'$, $m\angle F = m\angle F'$

Because a rotation is a rigid motion, the Composition Theorem (Theorem 4.1) guarantees that compositions of rotations and other rigid motions, such as translations and reflections, are rigid motions.

EXAMPLE 3 Performing a Composition

Graph $\overline{RS}$ with endpoints $R(1, -3)$ and $S(2, -6)$ and its image after the composition.

Reflection: in the y-axis

Rotation: $90°$ about the origin

COMMON ERROR

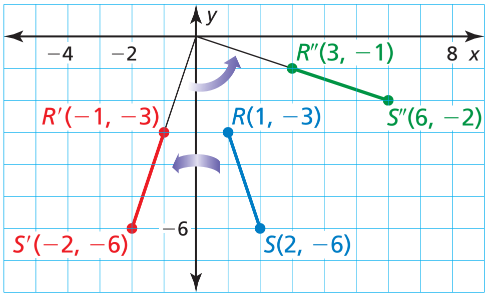
Unless you are told otherwise, perform the transformations in the order given.

SOLUTION

Step 1 Graph $\overline{RS}$.

Step 2 Reflect $\overline{RS}$ in the y-axis. $\overline{R'S'}$ has endpoints $R'(-1, -3)$ and $S'(-2, -6)$.

Step 3 Rotate $\overline{R'S'}$ $90°$ about the origin. $\overline{R''S''}$ has endpoints $R''(3, -1)$ and $S''(6, -2)$.



Monitoring Progress

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3. Graph $\overline{RS}$ from Example 3. Perform the rotation first, followed by the reflection. Does the order of the transformations matter? Explain.

4. **WHAT IF?** In Example 3, $\overline{RS}$ is reflected in the x-axis and rotated $180°$ about the origin. Graph $\overline{RS}$ and its image after the composition.

5. Graph $\overline{AB}$ with endpoints $A(-4, 4)$ and $B(-1, 7)$ and its image after the composition.

Translation: $(x, y) \rightarrow (x - 2, y - 1)$

Rotation: $90°$ about the origin

6. Graph $\triangle TUV$ with vertices $T(1, 2)$, $U(3, 5)$, and $V(6, 3)$ and its image after the composition.

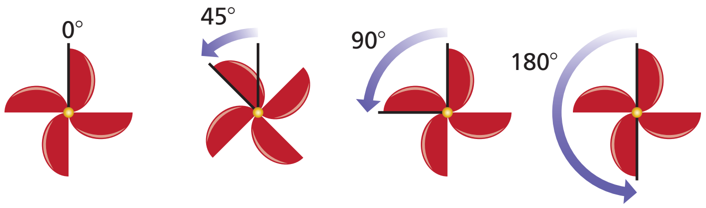
Rotation: $180°$ about the origin

Reflection: in the x-axis

Identifying Rotational Symmetry

A figure in the plane has **rotational symmetry** when the figure can be mapped onto itself by a rotation of 180° or less about the center of the figure. This point is the **center of symmetry**. Note that the rotation can be either clockwise or counterclockwise.

For example, the figure below has rotational symmetry, because a rotation of either 90° or 180° maps the figure onto itself (although a rotation of 45° does not).

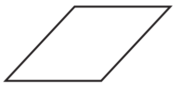


The figure above also has *point symmetry*, which is 180° rotational symmetry.

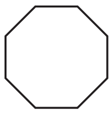
EXAMPLE 4 Identifying Rotational Symmetry

Does the figure have rotational symmetry? If so, describe any rotations that map the figure onto itself.

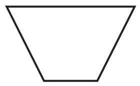
a. parallelogram



b. regular octagon

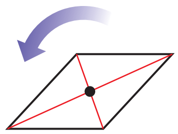


c. trapezoid

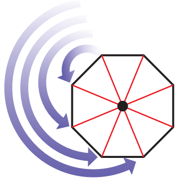


SOLUTION

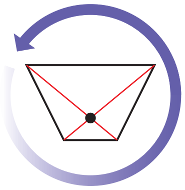
a. The parallelogram has rotational symmetry. The center is the intersection of the diagonals. A 180° rotation about the center maps the parallelogram onto itself.



b. The regular octagon has rotational symmetry. The center is the intersection of the diagonals. Rotations of 45°, 90°, 135°, or 180° about the center all map the octagon onto itself.



c. The trapezoid does not have rotational symmetry because no rotation of 180° or less maps the trapezoid onto itself.



Monitoring Progress

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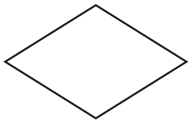
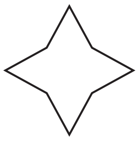
Determine whether the figure has rotational symmetry. If so, describe any rotations that map the figure onto itself.

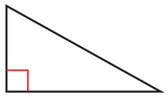
7. rhombus



8. octagon



9. right triangle



4.3 Exercises

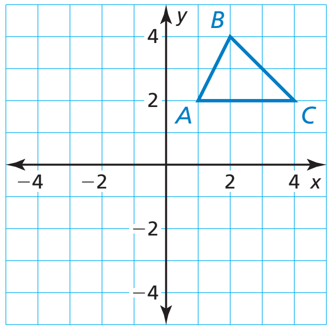
Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** When a point (a, b) is rotated counterclockwise about the origin, $(a, b) \rightarrow (b, -a)$ is the result of a rotation of ______.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

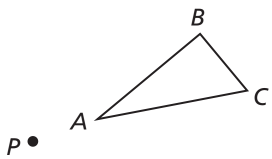
What are the coordinates of the vertices of the image after a 90° counterclockwise rotation about the origin?
What are the coordinates of the vertices of the image after a 270° clockwise rotation about the origin?
What are the coordinates of the vertices of the image after turning the figure 90° to the left about the origin?
What are the coordinates of the vertices of the image after a 270° counterclockwise rotation about the origin?



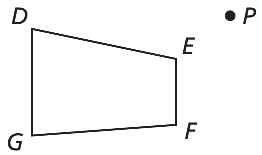
Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, trace the polygon and point *P*. Then draw a rotation of the polygon about point *P* using the given number of degrees. *(See Example 1.)*

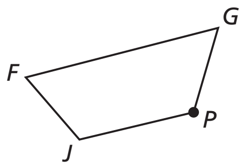
3. 30°



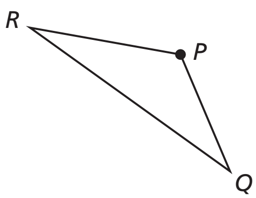
4. 80°



5. 150°

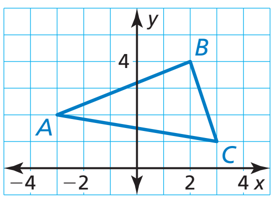


6. 130°

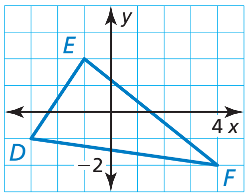


In Exercises 7–10, graph the polygon and its image after a rotation of the given number of degrees about the origin. *(See Example 2.)*

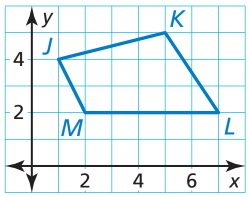
7. 90°



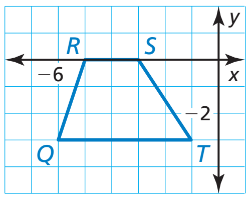
8. 180°



9. 180°



10. 270°



In Exercises 11–14, graph $\overline{XY}$ with endpoints $X(-3, 1)$ and $Y(4, -5)$ and its image after the composition. *(See Example 3.)*

11. **Translation:** $(x, y) \rightarrow (x, y + 2)$
Rotation: 90° about the origin

12. **Rotation:** 180° about the origin
Translation: $(x, y) \rightarrow (x - 1, y + 1)$

13. **Rotation:** 270° about the origin
Reflection: in the y-axis

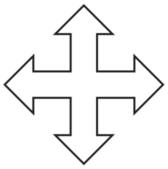
14. **Reflection:** in the line $y = x$
Rotation: 180° about the origin

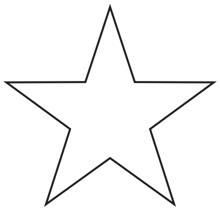
In Exercises 15 and 16, graph △*LMN* with vertices *L*(1, 6), *M*(−2, 4), and *N*(3, 2) and its image after the composition. *(See Example 3.)*

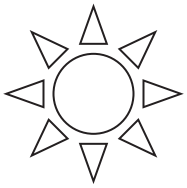
15. **Rotation:** 90° about the origin
Translation: $(x, y) \rightarrow (x - 3, y + 2)$

16. **Reflection:** in the *x*-axis
Rotation: 270° about the origin

In Exercises 17–20, determine whether the figure has rotational symmetry. If so, describe any rotations that map the figure onto itself. *(See Example 4.)*

17.


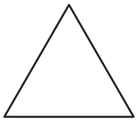
18.


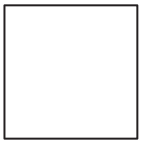
19.


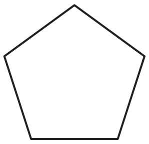
20.

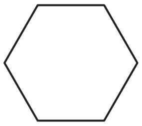

REPEATED REASONING **In Exercises 21–24, select the angles of rotational symmetry for the regular polygon. Select all that apply.**

Ⓐ 30°	Ⓑ 45°	Ⓒ 60°	Ⓓ 72°
Ⓔ 90°	Ⓕ 120°	Ⓖ 144°	Ⓗ 180°

21.


22.


23.


24.


ERROR ANALYSIS **In Exercises 25 and 26, the endpoints of $\overline{CD}$ are *C*(−1, 1) and *D*(2, 3). Describe and correct the error in finding the coordinates of the vertices of the image after a rotation of 270° about the origin.**

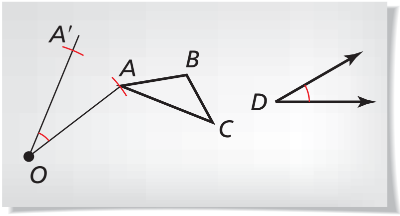
25.

✗ $C(-1, 1) \rightarrow C'(-1, -1)$
$D(2, 3) \rightarrow D'(2, -3)$

26.

✗ $C(-1, 1) \rightarrow C'(1, -1)$
$D(2, 3) \rightarrow D'(3, 2)$

27. **CONSTRUCTION** Follow these steps to construct a rotation of △*ABC* by angle *D* around a point *O*. Use a compass and straightedge.



Step 1 Draw △*ABC*, ∠*D*, and *O*, the center of rotation.

Step 2 Draw $\overline{OA}$. Use the construction for copying an angle to copy ∠*D* at *O*, as shown. Then use distance *OA* and center *O* to find *A*′.

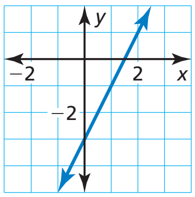
Step 3 Repeat Step 2 to find points *B*′ and *C*′. Draw △*A*′*B*′*C*′.

28. **REASONING** You enter the revolving door at a hotel.

a. You rotate the door 180°. What does this mean in the context of the situation? Explain.

b. You rotate the door 360°. What does this mean in the context of the situation? Explain.

29. **MATHEMATICAL CONNECTIONS** Use the graph of $y = 2x - 3$.



a. Rotate the line 90°, 180°, 270°, and 360° about the origin. Write the equation of the line for each image. Describe the relationship between the equation of the preimage and the equation of each image.

b. Do you think that the relationships you described in part (a) are true for any line? Explain your reasoning.

30. **MAKING AN ARGUMENT** Your friend claims that rotating a figure by 180° is the same as reflecting a figure in the *y*-axis and then reflecting it in the *x*-axis. Is your friend correct? Explain your reasoning.

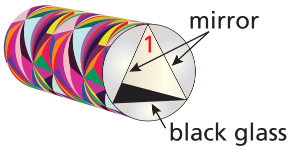
31. DRAWING CONCLUSIONS A figure only has point symmetry. How many times can you rotate the figure before it is back where it started?

32. ANALYZING RELATIONSHIPS Is it possible for a figure to have 90° rotational symmetry but not 180° rotational symmetry? Explain your reasoning.

33. ANALYZING RELATIONSHIPS Is it possible for a figure to have 180° rotational symmetry but not 90° rotational symmetry? Explain your reasoning.

34. THOUGHT PROVOKING Can rotations of 90°, 180°, 270°, and 360° be written as the composition of two reflections? Justify your answer.

35. USING AN EQUATION Inside a kaleidoscope, two mirrors are placed next to each other to form a V. The angle between the mirrors determines the number of lines of symmetry in the image. Use the formula $n(m\angle 1) = 180°$ to find the measure of $\angle 1$, the angle between the mirrors, for the number n of lines of symmetry.



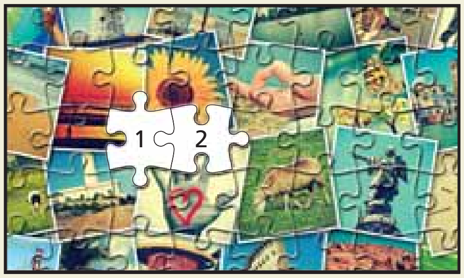
a.

b.

36. REASONING Use the coordinate rules for counterclockwise rotations about the origin to write coordinate rules for clockwise rotations of 90°, 180°, or 270° about the origin.

37. USING STRUCTURE $\triangle XYZ$ has vertices $X(2, 5)$, $Y(3, 1)$, and $Z(0, 2)$. Rotate $\triangle XYZ$ 90° about the point $P(-2, -1)$.

38. HOW DO YOU SEE IT? You are finishing the puzzle. The remaining two pieces both have rotational symmetry.

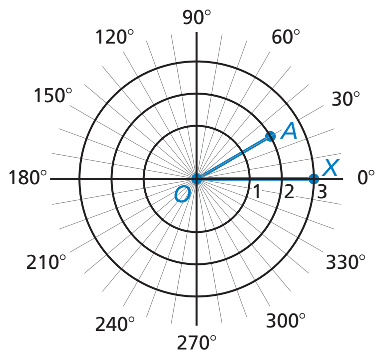


a. Describe the rotational symmetry of Piece 1 and of Piece 2.

b. You pick up Piece 1. How many different ways can it fit in the puzzle?

c. Before putting Piece 1 into the puzzle, you connect it to Piece 2. Now how many ways can it fit in the puzzle? Explain.

39. USING STRUCTURE A polar coordinate system locates a point in a plane by its distance from the origin O and by the measure of an angle with its vertex at the origin. For example, the point $A(2, 30°)$ is 2 units from the origin and $m\angle XOA = 30°$. What are the polar coordinates of the image of point A after a 90° rotation? a 180° rotation? a 270° rotation? Explain.

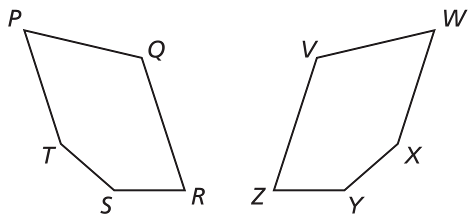


Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

The figures are congruent. Name the corresponding angles and the corresponding sides. *(Skills Review Handbook)*

40.



41.