4.4 Congruence and Transformations

Essential Question What conjectures can you make about a figure reflected in two lines?

EXPLORATION 1 Reflections in Parallel Lines

Work with a partner. Use dynamic geometry software to draw any scalene triangle and label it $\triangle ABC$.

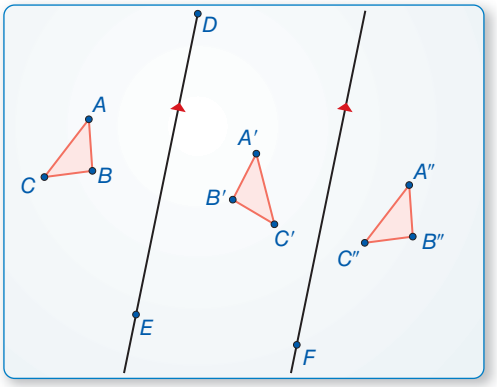
a. Draw any line $\overleftrightarrow{DE}$. Reflect $\triangle ABC$ in $\overleftrightarrow{DE}$ to form $\triangle A'B'C'$.

b. Draw a line parallel to $\overleftrightarrow{DE}$. Reflect $\triangle A'B'C'$ in the new line to form $\triangle A''B''C''$.

c. Draw the line through point A that is perpendicular to $\overleftrightarrow{DE}$. What do you notice?

d. Find the distance between points A and A''. Find the distance between the two parallel lines. What do you notice?

Sample



CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to make conjectures and justify your conclusions.

e. Hide $\triangle A'B'C'$. Is there a single transformation that maps $\triangle ABC$ to $\triangle A''B''C''$? Explain.

f. Make conjectures based on your answers in parts (c)–(e). Test your conjectures by changing $\triangle ABC$ and the parallel lines.

EXPLORATION 2 Reflections in Intersecting Lines

Work with a partner. Use dynamic geometry software to draw any scalene triangle and label it $\triangle ABC$.

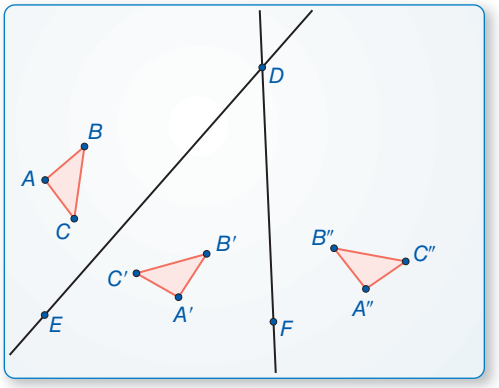
a. Draw any line $\overleftrightarrow{DE}$. Reflect $\triangle ABC$ in $\overleftrightarrow{DE}$ to form $\triangle A'B'C'$.

b. Draw any line $\overleftrightarrow{DF}$ so that angle EDF is less than or equal to 90°. Reflect $\triangle A'B'C'$ in $\overleftrightarrow{DF}$ to form $\triangle A''B''C''$.

c. Find the measure of $\angle EDF$. Rotate $\triangle ABC$ counterclockwise about point D using an angle twice the measure of $\angle EDF$.

d. Make a conjecture about a figure reflected in two intersecting lines. Test your conjecture by changing $\triangle ABC$ and the lines.

Sample



Communicate Your Answer

3. What conjectures can you make about a figure reflected in two lines?

4. Point Q is reflected in two parallel lines, $\overleftrightarrow{GH}$ and $\overleftrightarrow{JK}$, to form Q' and Q''. The distance from $\overleftrightarrow{GH}$ to $\overleftrightarrow{JK}$ is 3.2 inches. What is the distance QQ''?

4.4 Lesson

Core Vocabulary

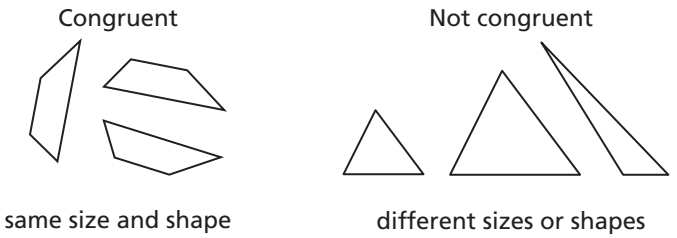
congruent figures, *p. 200*
congruence transformation, *p. 201*

What You Will Learn

- Identify congruent figures.
- Describe congruence transformations.
- Use theorems about congruence transformations.

Identifying Congruent Figures

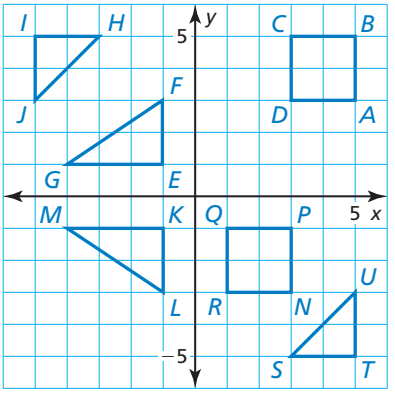
Two geometric figures are **congruent figures** if and only if there is a rigid motion or a composition of rigid motions that maps one of the figures onto the other. Congruent figures have the same size and shape.



You can identify congruent figures in the coordinate plane by identifying the rigid motion or composition of rigid motions that maps one of the figures onto the other. Recall from Postulates 4.1–4.3 and Theorem 4.1 that translations, reflections, rotations, and compositions of these transformations are rigid motions.

EXAMPLE 1 Identifying Congruent Figures

Identify any congruent figures in the coordinate plane. Explain.



SOLUTION

Square *NPQR* is a translation of square *ABCD* 2 units left and 6 units down. So, square *ABCD* and square *NPQR* are congruent.

$\triangle KLM$ is a reflection of $\triangle EFG$ in the x-axis. So, $\triangle EFG$ and $\triangle KLM$ are congruent.

$\triangle STU$ is a 180° rotation of $\triangle HIJ$. So, $\triangle HIJ$ and $\triangle STU$ are congruent.

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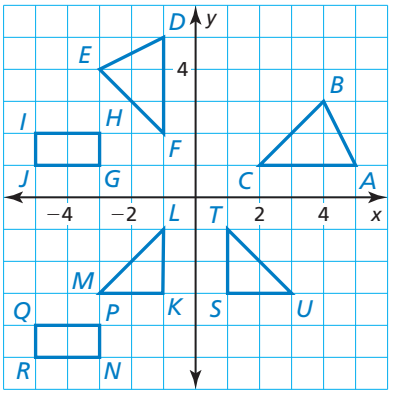
1. Identify any congruent figures in the coordinate plane. Explain.



Congruence Transformations

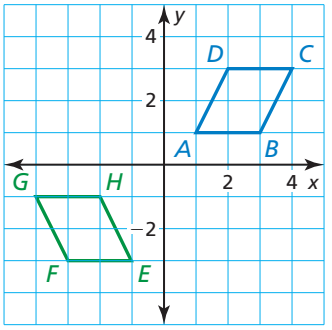
Another name for a rigid motion or a combination of rigid motions is a **congruence transformation** because the preimage and image are congruent. The terms "rigid motion" and "congruence transformation" are interchangeable.

READING

You can read the notation ▱*ABCD* as "parallelogram *A, B, C, D*."

EXAMPLE 2 Describing a Congruence Transformation

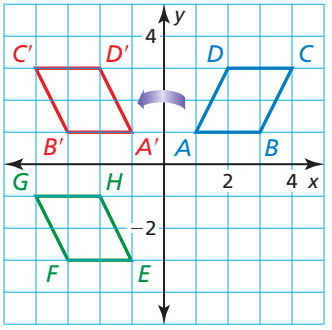
Describe a congruence transformation that maps ▱*ABCD* to ▱*EFGH*.



SOLUTION

The two vertical sides of ▱*ABCD* rise from left to right, and the two vertical sides of ▱*EFGH* fall from left to right. If you reflect ▱*ABCD* in the *y*-axis, as shown, then the image, ▱*A′B′C′D′*, will have the same orientation as ▱*EFGH*.

Then you can map ▱*A′B′C′D′* to ▱*EFGH* using a translation of 4 units down.



▶ So, a congruence transformation that maps ▱*ABCD* to ▱*EFGH* is a reflection in the *y*-axis followed by a translation of 4 units down.

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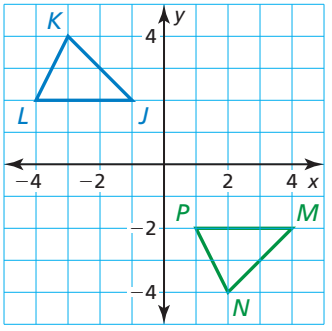
2. In Example 2, describe another congruence transformation that maps ▱*ABCD* to ▱*EFGH*.

3. Describe a congruence transformation that maps △*JKL* to △*MNP*.



Using Theorems about Congruence Transformations

Compositions of two reflections result in either a translation or a rotation. A composition of two reflections in parallel lines results in a translation, as described in the following theorem.

Theorem

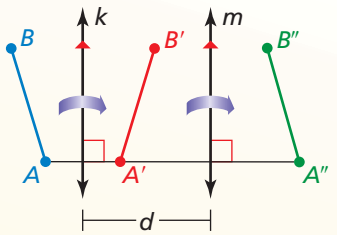
Theorem 4.2 Reflections in Parallel Lines Theorem

If lines k and m are parallel, then a reflection in line k followed by a reflection in line m is the same as a translation.

If A'' is the image of A, then

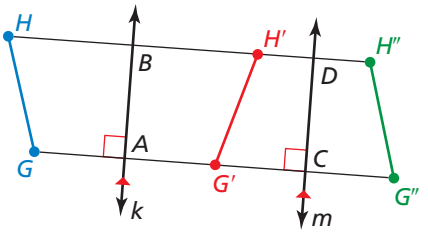
1. $\overline{AA''}$ is perpendicular to k and m, and
2. $AA'' = 2d$, where d is the distance between k and m.

Proof Ex. 31, p. 206



EXAMPLE 3 Using the Reflections in Parallel Lines Theorem

In the diagram, a reflection in line k maps $\overline{GH}$ to $\overline{G'H'}$. A reflection in line m maps $\overline{G'H'}$ to $\overline{G''H''}$. Also, $HB = 9$ and $DH'' = 4$.



a. Name any segments congruent to each segment: $\overline{GH}$, $\overline{HB}$, and $\overline{GA}$.

b. Does $AC = BD$? Explain.

c. What is the length of $\overline{GG''}$?

SOLUTION

a. $\overline{GH} \cong \overline{G'H'}$, and $\overline{GH} \cong \overline{G''H''}$. $\overline{HB} \cong \overline{H'B}$. $\overline{GA} \cong \overline{G'A}$.

b. Yes, $AC = BD$ because $\overline{GG''}$ and $\overline{HH''}$ are perpendicular to both k and m. So, $\overline{BD}$ and $\overline{AC}$ are opposite sides of a rectangle.

c. By the properties of reflections, $H'B = 9$ and $H'D = 4$. The Reflections in Parallel Lines Theorem implies that $GG'' = HH'' = 2 \cdot BD$, so the length of $\overline{GG''}$ is $2(9 + 4) = 26$ units.

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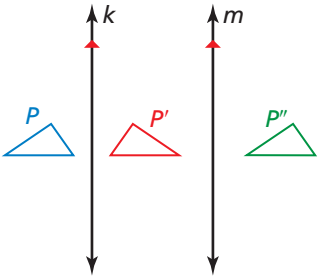
Use the figure. The distance between line k and line m is 1.6 centimeters.



4. The preimage is reflected in line k, then in line m. Describe a single transformation that maps the blue figure to the green figure.
5. What is the relationship between $\overline{PP'}$ and line k? Explain.
6. What is the distance between P and P''?

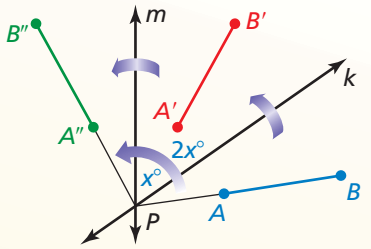
A composition of two reflections in intersecting lines results in a rotation, as described in the following theorem.

Theorem

Theorem 4.3 Reflections in Intersecting Lines Theorem

If lines k and m intersect at point P, then a reflection in line k followed by a reflection in line m is the same as a rotation about point P.

The angle of rotation is $2x°$, where $x°$ is the measure of the acute or right angle formed by lines k and m.

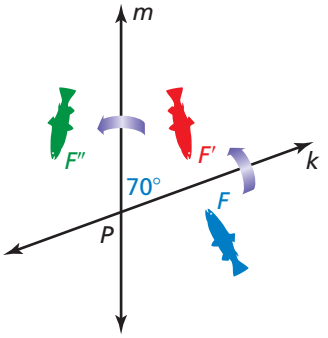


$m\angle BPB'' = 2x°$

Proof Ex. 31, p. 250

EXAMPLE 4 Using the Reflections in Intersecting Lines Theorem

In the diagram, the figure is reflected in line k. The image is then reflected in line m. Describe a single transformation that maps F to F''.



SOLUTION

By the Reflections in Intersecting Lines Theorem, a reflection in line k followed by a reflection in line m is the same as a rotation about point P. The measure of the acute angle formed between lines k and m is 70°. So, by the Reflections in Intersecting Lines Theorem, the angle of rotation is $2(70°) = 140°$. A single transformation that maps F to F'' is a 140° rotation about point P.

▶ You can check that this is correct by tracing lines k and m and point F, then rotating the point 140°.

Monitoring Progress

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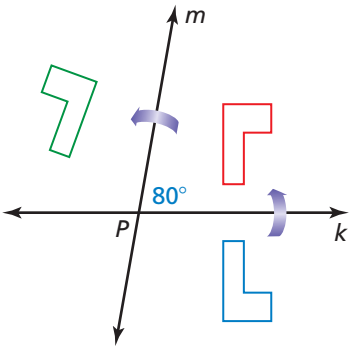
7. In the diagram, the preimage is reflected in line k, then in line m. Describe a single transformation that maps the blue figure onto the green figure.

8. A rotation of 76° maps C to C'. To map C to C' using two reflections, what is the measure of the angle formed by the intersecting lines of reflection?



4.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

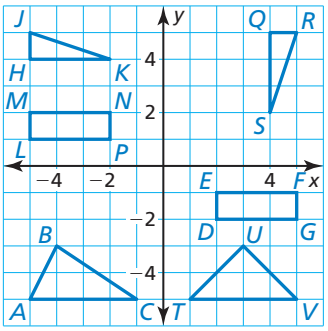
1. **COMPLETE THE SENTENCE** Two geometric figures are _____ if and only if there is a rigid motion or a composition of rigid motions that moves one of the figures onto the other.

2. **VOCABULARY** Why is the term *congruence transformation* used to refer to a rigid motion?

Monitoring Progress and Modeling with Mathematics

In Exercises 3 and 4, identify any congruent figures in the coordinate plane. Explain. *(See Example 1.)*

3.



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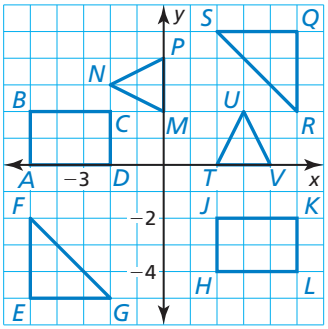
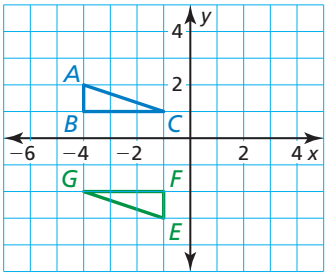
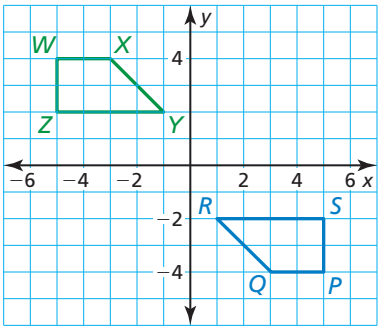


In Exercises 5 and 6, describe a congruence transformation that maps the blue preimage to the green image. *(See Example 2.)*

5.



6.



In Exercises 7–10, determine whether the polygons with the given vertices are congruent. Use transformations to explain your reasoning.

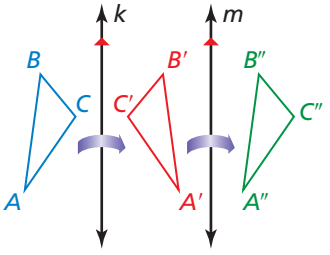
7. $Q(2, 4)$, $R(5, 4)$, $S(4, 1)$ and $T(6, 4)$, $U(9, 4)$, $V(8, 1)$

8. $W(-3, 1)$, $X(2, 1)$, $Y(4, -4)$, $Z(-5, -4)$ and $C(-1, -3)$, $D(-1, 2)$, $E(4, 4)$, $F(4, -5)$

9. $J(1, 1)$, $K(3, 2)$, $L(4, 1)$ and $M(6, 1)$, $N(5, 2)$, $P(2, 1)$

10. $A(0, 0)$, $B(1, 2)$, $C(4, 2)$, $D(3, 0)$ and $E(0, -5)$, $F(-1, -3)$, $G(-4, -3)$, $H(-3, -5)$

In Exercises 11–14, $k \parallel m$, $\triangle ABC$ is reflected in line k, and $\triangle A'B'C'$ is reflected in line m. *(See Example 3.)*



11. A translation maps $\triangle ABC$ onto which triangle?

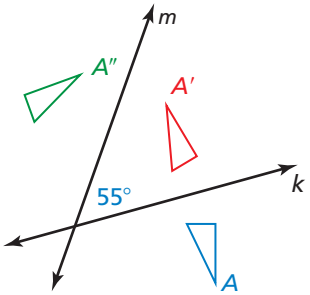
12. Which lines are perpendicular to $\overline{AA''}$?

13. If the distance between k and m is 2.6 inches, what is the length of $\overline{CC''}$?

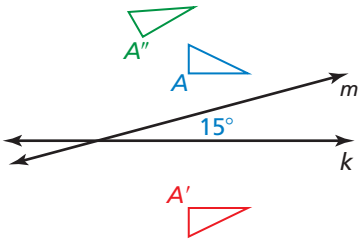
14. Is the distance from B' to m the same as the distance from B'' to m? Explain.

In Exercises 15 and 16, find the angle of rotation that maps A onto A''. *(See Example 4.)*

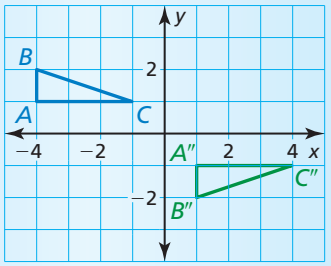
15.



16.

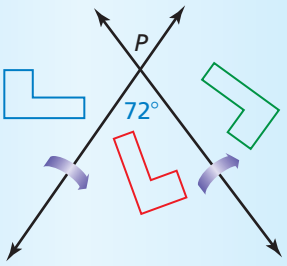


17. **ERROR ANALYSIS** Describe and correct the error in describing the congruence transformation.



$\triangle ABC$ is mapped to $\triangle A''B''C''$ by a translation 3 units down and a reflection in the y-axis.

18. **ERROR ANALYSIS** Describe and correct the error in using the Reflections in Intersecting Lines Theorem (Theorem 4.3).



A 72° rotation about point P maps the blue image to the green image.

In Exercises 19–22, find the measure of the acute or right angle formed by intersecting lines so that C can be mapped to C' using two reflections.

19. A rotation of 84° maps C to C'.

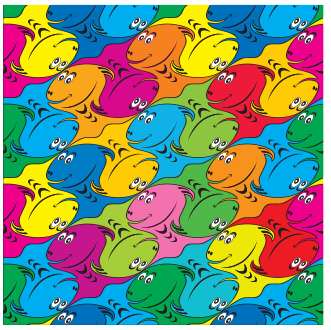
20. A rotation of 24° maps C to C'.

21. The rotation $(x, y) \rightarrow (-x, -y)$ maps C to C'.

22. The rotation $(x, y) \rightarrow (y, -x)$ maps C to C'.

23. **REASONING** Use the Reflections in Parallel Lines Theorem (Theorem 4.2) to explain how you can make a glide reflection using three reflections. How are the lines of reflection related?

24. **DRAWING CONCLUSIONS** The pattern shown is called a *tessellation*.



a. What transformations did the artist use when creating this tessellation?

b. Are the individual figures in the tessellation congruent? Explain your reasoning.

CRITICAL THINKING **In Exercises 25–28, tell whether the statement is *always*, *sometimes*, or *never* true. Explain your reasoning.**

25. A congruence transformation changes the size of a figure.

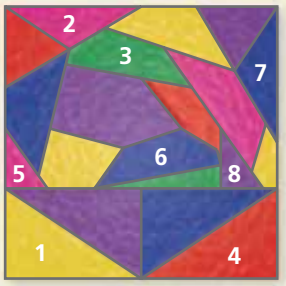
26. If two figures are congruent, then there is a rigid motion or a composition of rigid motions that maps one figure onto the other.

27. The composition of two reflections results in the same image as a rotation.

28. A translation results in the same image as the composition of two reflections.

29. **REASONING** During a presentation, a marketing representative uses a projector so everyone in the auditorium can view the advertisement. Is this projection a congruence transformation? Explain your reasoning.

30. HOW DO YOU SEE IT? What type of congruence transformation can be used to verify each statement about the stained glass window?



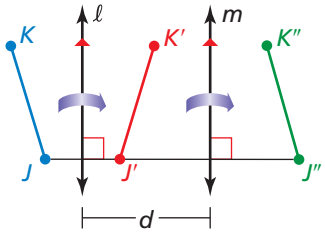
a. Triangle 5 is congruent to Triangle 8.

b. Triangle 1 is congruent to Triangle 4.

c. Triangle 2 is congruent to Triangle 7.

d. Pentagon 3 is congruent to Pentagon 6.

31. PROVING A THEOREM Prove the Reflections in Parallel Lines Theorem (Theorem 4.2).



Given A reflection in line ℓ maps $\overline{JK}$ to $\overline{J'K'}$, a reflection in line m maps $\overline{J'K'}$ to $\overline{J''K''}$, and $\ell \parallel m$.

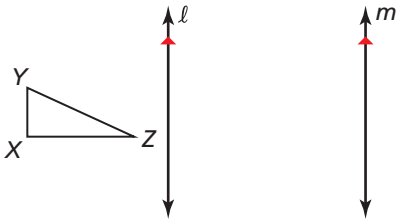
Prove **a.** $\overline{KK''}$ is perpendicular to ℓ and m.

b. $KK'' = 2d$, where d is the distance between ℓ and m.

32. THOUGHT PROVOKING A *tessellation* is the covering of a plane with congruent figures so that there are no gaps or overlaps (see Exercise 24). Draw a tessellation that involves two or more types of transformations. Describe the transformations that are used to create the tessellation.

33. MAKING AN ARGUMENT $\overline{PQ}$, with endpoints $P(1, 3)$ and $Q(3, 2)$, is reflected in the y-axis. The image $\overline{P'Q'}$ is then reflected in the x-axis to produce the image $\overline{P''Q''}$. One classmate says that $\overline{PQ}$ is mapped to $\overline{P''Q''}$ by the translation $(x, y) \rightarrow (x - 4, y - 5)$. Another classmate says that $\overline{PQ}$ is mapped to $\overline{P''Q''}$ by a $(2 \cdot 90)°$, or $180°$, rotation about the origin. Which classmate is correct? Explain your reasoning.

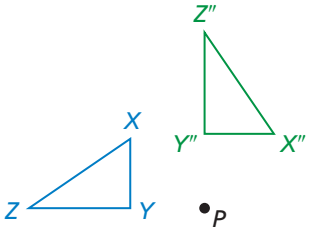
34. CRITICAL THINKING Does the order of reflections for a composition of two reflections in parallel lines matter? For example, is reflecting $\triangle XYZ$ in line ℓ and then its image in line m the same as reflecting $\triangle XYZ$ in line m and then its image in line ℓ?



CONSTRUCTION **In Exercises 35 and 36, copy the figure. Then use a compass and straightedge to construct two lines of reflection that produce a composition of reflections resulting in the same image as the given transformation.**

35. Translation: $\triangle ABC \rightarrow \triangle A''B''C''$

36. Rotation about *P*: $\triangle XYZ \rightarrow \triangle X''Y''Z''$



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the equation. Check your solution. *(Skills Review Handbook)*

37. $5x + 16 = -3x$

38. $12 + 6m = 2m$

39. $4b + 8 = 6b - 4$

40. $7w - 9 = 13 - 4w$

41. $7(2n + 11) = 4n$

42. $-2(8 - y) = -6y$

43. Last year, the track team's yard sale earned \$500. This year, the yard sale earned \$625. What is the percent of increase? *(Skills Review Handbook)*