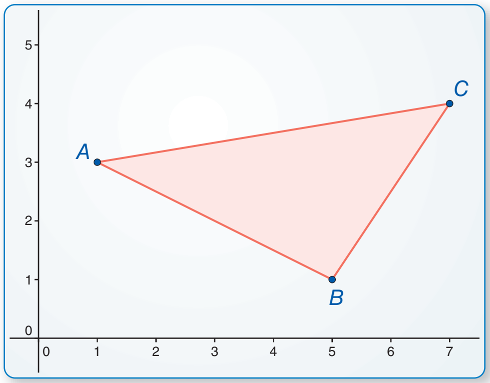
6.5 Indirect Proof and Inequalities in One Triangle

Essential Question How are the sides related to the angles of a triangle? How are any two sides of a triangle related to the third side?

EXPLORATION 1 Comparing Angle Measures and Side Lengths

Work with a partner. Use dynamic geometry software. Draw any scalene $\triangle ABC$.

a. Find the side lengths and angle measures of the triangle.



Sample

Points	Angles
$A(1, 3)$	$m\angle A = ?$
$B(5, 1)$	$m\angle B = ?$
$C(7, 4)$	$m\angle C = ?$

Segments
$BC = ?$
$AC = ?$
$AB = ?$

b. Order the side lengths. Order the angle measures. What do you observe?

c. Drag the vertices of $\triangle ABC$ to form new triangles. Record the side lengths and angle measures in a table. Write a conjecture about your findings.

ATTENDING TO PRECISION

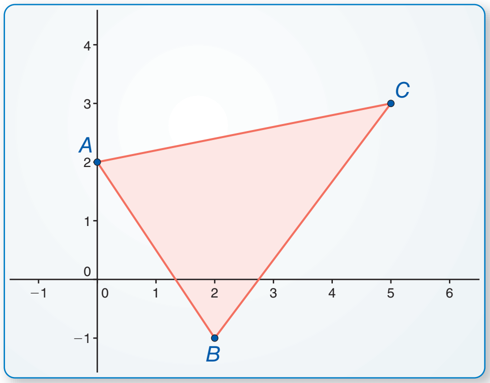
To be proficient in math, you need to express numerical answers with a degree of precision appropriate for the content.

EXPLORATION 2 A Relationship of the Side Lengths of a Triangle

Work with a partner. Use dynamic geometry software. Draw any $\triangle ABC$.

a. Find the side lengths of the triangle.

b. Compare each side length with the sum of the other two side lengths.



Sample

Points
$A(0, 2)$
$B(2, -1)$
$C(5, 3)$

Segments
$BC = ?$
$AC = ?$
$AB = ?$

c. Drag the vertices of $\triangle ABC$ to form new triangles and repeat parts (a) and (b). Organize your results in a table. Write a conjecture about your findings.

Communicate Your Answer

3. How are the sides related to the angles of a triangle? How are any two sides of a triangle related to the third side?

4. Is it possible for a triangle to have side lengths of 3, 4, and 10? Explain.

6.5 Lesson

Core Vocabulary

indirect proof, *p. 336*

Previous
proof
inequality

What You Will Learn

- Write indirect proofs.
- List sides and angles of a triangle in order by size.
- Use the Triangle Inequality Theorem to find possible side lengths of triangles.

Writing an Indirect Proof

Suppose a student looks around the cafeteria, concludes that hamburgers are not being served, and explains as follows.

> *At first, I assumed that we are having hamburgers because today is Tuesday, and Tuesday is usually hamburger day.*
>
> *There is always ketchup on the table when we have hamburgers, so I looked for the ketchup, but I didn't see any.*
>
> *So, my assumption that we are having hamburgers must be false.*

The student uses *indirect* reasoning. In an **indirect proof**, you start by making the temporary assumption that the desired conclusion is false. By then showing that this assumption leads to a logical impossibility, you prove the original statement true *by contradiction*.

Core Concept

How to Write an Indirect Proof (Proof by Contradiction)

Step 1 Identify the statement you want to prove. Assume temporarily that this statement is false by assuming that its opposite is true.

Step 2 Reason logically until you reach a contradiction.

Step 3 Point out that the desired conclusion must be true because the contradiction proves the temporary assumption false.

EXAMPLE 1 Writing an Indirect Proof

Write an indirect proof that in a given triangle, there can be at most one right angle.

Given $\triangle ABC$

Prove $\triangle ABC$ can have at most one right angle.

SOLUTION

Step 1 Assume temporarily that $\triangle ABC$ has two right angles. Then assume $\angle A$ and $\angle B$ are right angles.

Step 2 By the definition of right angle, $m\angle A = m\angle B = 90^\circ$. By the Triangle Sum Theorem (Theorem 5.1), $m\angle A + m\angle B + m\angle C = 180^\circ$. Using the Substitution Property of Equality, $90^\circ + 90^\circ + m\angle C = 180^\circ$. So, $m\angle C = 0^\circ$ by the Subtraction Property of Equality. A triangle cannot have an angle measure of 0°. So, this contradicts the given information.

Step 3 So, the assumption that $\triangle ABC$ has two right angles must be false, which proves that $\triangle ABC$ can have at most one right angle.

READING

You have reached a *contradiction* when you have two statements that cannot both be true at the same time.

Monitoring Progress

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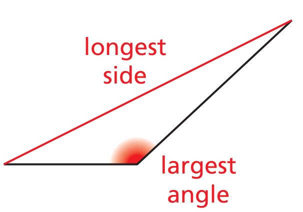
1. Write an indirect proof that a scalene triangle cannot have two congruent angles.

Relating Sides and Angles of a Triangle

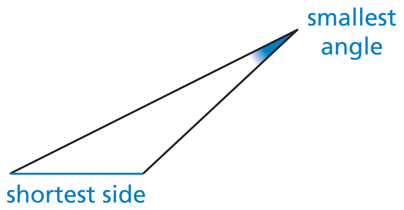
EXAMPLE 2 Relating Side Length and Angle Measure

Draw an obtuse scalene triangle. Find the largest angle and longest side and mark them in red. Find the smallest angle and shortest side and mark them in blue. What do you notice?

SOLUTION



The longest side and largest angle are opposite each other.



The shortest side and smallest angle are opposite each other.

COMMON ERROR

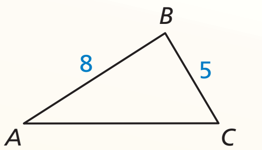
Be careful not to confuse the symbol $\angle$ meaning *angle* with the symbol $<$ meaning *is less than*. Notice that the bottom edge of the angle symbol is horizontal.

The relationships in Example 2 are true for all triangles, as stated in the two theorems below. These relationships can help you decide whether a particular arrangement of side lengths and angle measures in a triangle may be possible.

Theorems

Theorem 6.9 Triangle Longer Side Theorem

If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

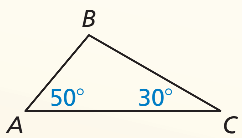


$AB > BC$, so $m\angle C > m\angle A$.

Proof Ex. 43, p. 342

Theorem 6.10 Triangle Larger Angle Theorem

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.



$m\angle A > m\angle C$, so $BC > AB$.

Proof p. 337

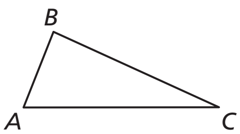
COMMON ERROR

Be sure to consider all cases when assuming the opposite is true.

PROOF Triangle Larger Angle Theorem

Given $m\angle A > m\angle C$

Prove $BC > AB$



Indirect Proof

Step 1 Assume temporarily that $BC \ngtr AB$. Then it follows that either $BC < AB$ or $BC = AB$.

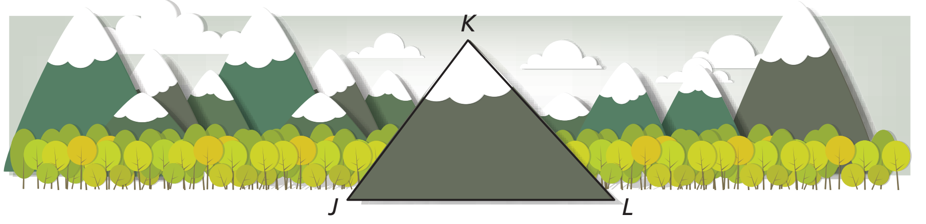
Step 2 If $BC < AB$, then $m\angle A < m\angle C$ by the Triangle Longer Side Theorem.

If $BC = AB$, then $m\angle A = m\angle C$ by the Base Angles Theorem (Thm. 5.6).

Step 3 Both conclusions contradict the given statement that $m\angle A > m\angle C$. So, the temporary assumption that $BC \ngtr AB$ cannot be true. This proves that $BC > AB$.

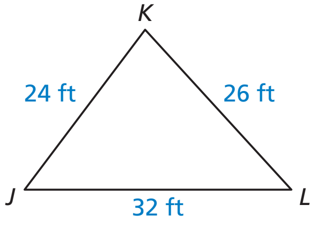
EXAMPLE 3 Ordering Angle Measures of a Triangle

You are constructing a stage prop that shows a large triangular mountain. The bottom edge of the mountain is about 32 feet long, the left slope is about 24 feet long, and the right slope is about 26 feet long. List the angles of $\triangle JKL$ in order from smallest to largest.



SOLUTION

Draw the triangle that represents the mountain. Label the side lengths.



The sides from shortest to longest are $\overline{JK}$, $\overline{KL}$, and $\overline{JL}$. The angles opposite these sides are $\angle L$, $\angle J$, and $\angle K$, respectively.

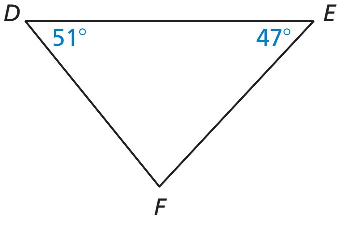
▶ So, by the Triangle Longer Side Theorem, the angles from smallest to largest are $\angle L$, $\angle J$, and $\angle K$.

EXAMPLE 4 Ordering Side Lengths of a Triangle

List the sides of $\triangle DEF$ in order from shortest to longest.

SOLUTION

First, find $m\angle F$ using the Triangle Sum Theorem (Theorem 5.1).



$$m\angle D + m\angle E + m\angle F = 180^\circ$$

$$51^\circ + 47^\circ + m\angle F = 180^\circ$$

$$m\angle F = 82^\circ$$

The angles from smallest to largest are $\angle E$, $\angle D$, and $\angle F$. The sides opposite these angles are $\overline{DF}$, $\overline{EF}$, and $\overline{DE}$, respectively.

▶ So, by the Triangle Larger Angle Theorem, the sides from shortest to longest are $\overline{DF}$, $\overline{EF}$, and $\overline{DE}$.

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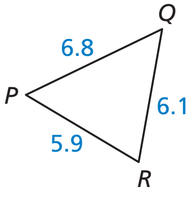
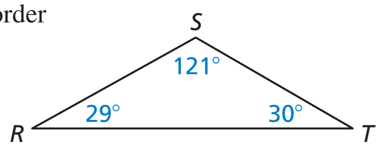
2. List the angles of $\triangle PQR$ in order from smallest to largest.

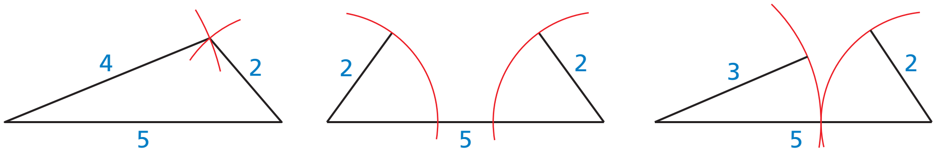


3. List the sides of $\triangle RST$ in order from shortest to longest.



Using the Triangle Inequality Theorem

Not every group of three segments can be used to form a triangle. The lengths of the segments must fit a certain relationship. For example, three attempted triangle constructions using segments with given lengths are shown below. Only the first group of segments forms a triangle.

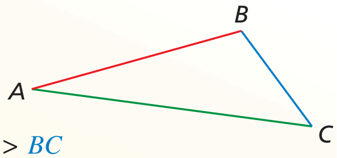


When you start with the longest side and attach the other two sides at its endpoints, you can see that the other two sides are not long enough to form a triangle in the second and third figures. This leads to the *Triangle Inequality Theorem*.

Theorem

Theorem 6.11 Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.



$AB + BC > AC$ $\quad$ $AC + BC > AB$ $\quad$ $AB + AC > BC$

Proof Ex. 47, p. 342

EXAMPLE 5 Finding Possible Side Lengths

A triangle has one side of length 14 and another side of length 9. Describe the possible lengths of the third side.

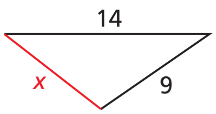
SOLUTION

Let x represent the length of the third side. Draw diagrams to help visualize the small and large values of x. Then use the Triangle Inequality Theorem to write and solve inequalities.

READING

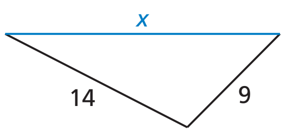
You can combine the two inequalities, $x > 5$ and $x < 23$, to write the compound inequality $5 < x < 23$. This can be read as *x is between 5 and 23.*

Small values of x



$$x + 9 > 14$$
$$x > 5$$

Large values of x



$$9 + 14 > x$$
$$23 > x, \text{ or } x < 23$$

▶ The length of the third side must be greater than 5 and less than 23.

Monitoring Progress

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4. A triangle has one side of length 12 inches and another side of length 20 inches. Describe the possible lengths of the third side.

Decide whether it is possible to construct a triangle with the given side lengths. Explain your reasoning.

5. 4 ft, 9 ft, 10 ft

6. 8 m, 9 m, 18 m

7. 5 cm, 7 cm, 12 cm

6.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** Why is an indirect proof also called a *proof by contradiction*?

2. **WRITING** How can you tell which side of a triangle is the longest from the angle measures of the triangle? How can you tell which side is the shortest?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, write the first step in an indirect proof of the statement. *(See Example 1.)*

3. If $WV + VU \neq 12$ inches and $VU = 5$ inches, then $WV \neq 7$ inches.

4. If x and y are odd integers, then xy is odd.

5. In $\triangle ABC$, if $m\angle A = 100°$, then $\angle B$ is not a right angle.

6. In $\triangle JKL$, if M is the midpoint of $\overline{KL}$, then $\overline{JM}$ is a median.

In Exercises 7 and 8, determine which two statements contradict each other. Explain your reasoning.

7. Ⓐ $\triangle LMN$ is a right triangle.
   Ⓑ $\angle L \cong \angle N$
   Ⓒ $\triangle LMN$ is equilateral.

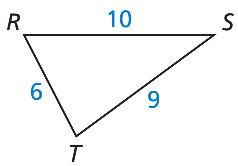
8. Ⓐ Both $\angle X$ and $\angle Y$ have measures greater than 20°.
   Ⓑ Both $\angle X$ and $\angle Y$ have measures less than 30°.
   Ⓒ $m\angle X + m\angle Y = 62°$

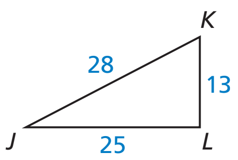
In Exercises 9 and 10, use a ruler and protractor to draw the given type of triangle. Mark the largest angle and longest side in red and the smallest angle and shortest side in blue. What do you notice? *(See Example 2.)*

9. acute scalene

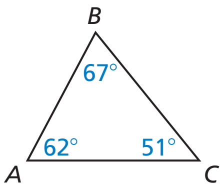
10. right scalene

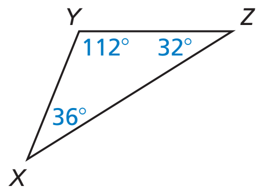
In Exercises 11 and 12, list the angles of the given triangle from smallest to largest. *(See Example 3.)*

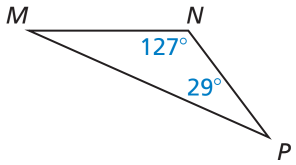
11.


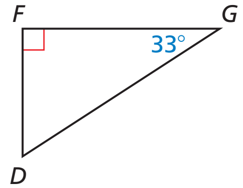
12.


In Exercises 13–16, list the sides of the given triangle from shortest to longest. *(See Example 4.)*

13.


14.


15.


16.


In Exercises 17–20, describe the possible lengths of the third side of the triangle given the lengths of the other two sides. *(See Example 5.)*

17. 5 inches, 12 inches

18. 12 feet, 18 feet

19. 2 feet, 40 inches

20. 25 meters, 25 meters

In Exercises 21–24, is it possible to construct a triangle with the given side lengths? If not, explain why not.

21. 6, 7, 11

22. 3, 6, 9

23. 28, 17, 46

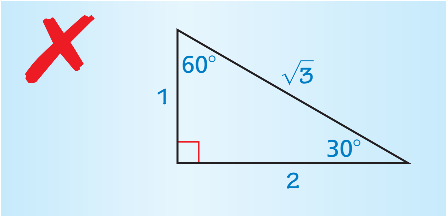
24. 35, 120, 125

25. **ERROR ANALYSIS** Describe and correct the error in writing the first step of an indirect proof.

    Show that $\angle A$ is obtuse.

    Step 1 Assume temporarily that $\angle A$ is acute.

26. ERROR ANALYSIS Describe and correct the error in labeling the side lengths 1, 2, and $\sqrt{3}$ on the triangle.



27. REASONING You are a lawyer representing a client who has been accused of a crime. The crime took place in Los Angeles, California. Security footage shows your client in New York at the time of the crime. Explain how to use indirect reasoning to prove your client is innocent.

28. REASONING Your class has fewer than 30 students. The teacher divides your class into two groups. The first group has 15 students. Use indirect reasoning to show that the second group must have fewer than 15 students.

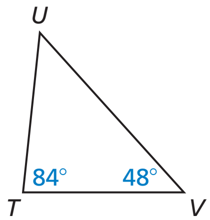
29. PROBLEM SOLVING Which statement about $\triangle TUV$ is false?

Ⓐ $UV > TU$

Ⓑ $UV + TV > TU$

Ⓒ $UV < TV$

Ⓓ $\triangle TUV$ is isosceles.



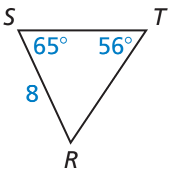
30. PROBLEM SOLVING In $\triangle RST$, which is a possible side length for ST? Select all that apply.

Ⓐ 7

Ⓑ 8

Ⓒ 9

Ⓓ 10



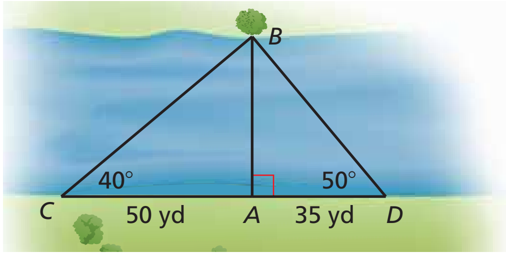
31. PROOF Write an indirect proof that an odd number is not divisible by 4.

32. PROOF Write an indirect proof of the statement "In $\triangle QRS$, if $m\angle Q + m\angle R = 90°$, then $m\angle S = 90°$."

33. WRITING Explain why the hypotenuse of a right triangle must always be longer than either leg.

34. CRITICAL THINKING Is it possible to decide if three side lengths form a triangle without checking all three inequalities shown in the Triangle Inequality Theorem (Theorem 6.11)? Explain your reasoning.

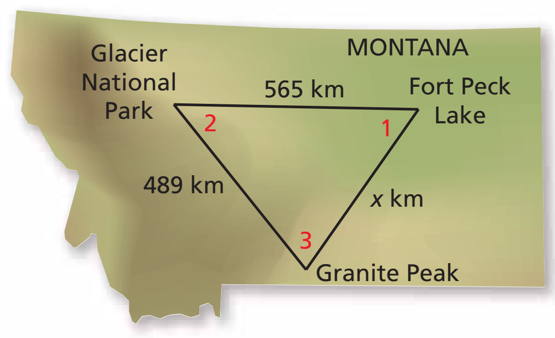
35. MODELING WITH MATHEMATICS You can estimate the width of the river from point A to the tree at point B by measuring the angle to the tree at several locations along the riverbank. The diagram shows the results for locations C and D.



a. Using $\triangle BCA$ and $\triangle BDA$, determine the possible widths of the river. Explain your reasoning.

b. What could you do if you wanted a closer estimate?

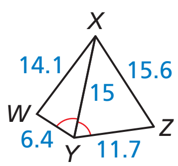
36. MODELING WITH MATHEMATICS You travel from Fort Peck Lake to Glacier National Park and from Glacier National Park to Granite Peak.



a. Write two inequalities to represent the possible distances from Granite Peak back to Fort Peck Lake.

b. How is your answer to part (a) affected if you know that $m\angle 2 < m\angle 1$ and $m\angle 2 < m\angle 3$?

37. REASONING In the figure, $\overline{XY}$ bisects $\angle WYZ$. List all six angles of $\triangle XYZ$ and $\triangle WXY$ in order from smallest to largest. Explain your reasoning.

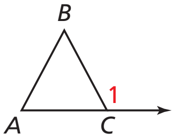


38. MATHEMATICAL CONNECTIONS In $\triangle DEF$, $m\angle D = (x + 25)°$, $m\angle E = (2x - 4)°$, and $m\angle F = 63°$. List the side lengths and angle measures of the triangle in order from least to greatest.

39. **ANALYZING RELATIONSHIPS** Another triangle inequality relationship is given by the Exterior Angle Inequality Theorem. It states:

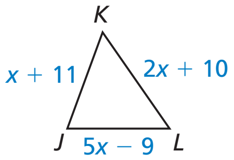
The measure of an exterior angle of a triangle is greater than the measure of either of the nonadjacent interior angles.

Explain how you know that $m\angle 1 > m\angle A$ and $m\angle 1 > m\angle B$ in $\triangle ABC$ with exterior angle $\angle 1$.

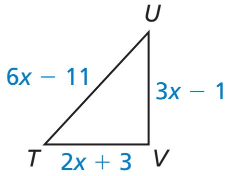


MATHEMATICAL CONNECTIONS **In Exercises 40 and 41, describe the possible values of *x*.**

40.

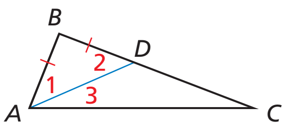


41.



42. **HOW DO YOU SEE IT?** Your house is on the corner of Hill Street and Eighth Street. The library is on the corner of View Street and Seventh Street. What is the shortest route to get from your house to the library? Explain your reasoning.

43. **PROVING A THEOREM** Use the diagram to prove the Triangle Longer Side Theorem (Theorem 6.9).



Given $BC > AB$, $BD = BA$

Prove $m\angle BAC > m\angle C$

44. **USING STRUCTURE** The length of the base of an isosceles triangle is ℓ. Describe the possible lengths for each leg. Explain your reasoning.

45. **MAKING AN ARGUMENT** Your classmate claims to have drawn a triangle with one side length of 13 inches and a perimeter of 2 feet. Is this possible? Explain your reasoning.

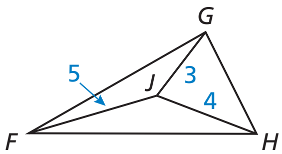
46. **THOUGHT PROVOKING** Cut two pieces of string that are each 24 centimeters long. Construct an isosceles triangle out of one string and a scalene triangle out of the other. Measure and record the side lengths. Then classify each triangle by its angles.

47. **PROVING A THEOREM** Prove the Triangle Inequality Theorem (Theorem 6.11).

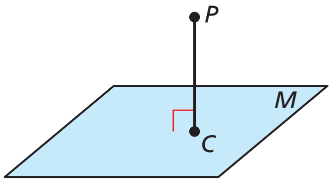
Given $\triangle ABC$

Prove $AB + BC > AC$, $AC + BC > AB$, and $AB + AC > BC$

48. **ATTENDING TO PRECISION** The perimeter of $\triangle HGF$ must be between what two integers? Explain your reasoning.



49. **PROOF** Write an indirect proof that a perpendicular segment is the shortest segment from a point to a plane.



Given $\overline{PC} \perp$ plane M

Prove $\overline{PC}$ is the shortest segment from P to plane M.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Name the included angle between the pair of sides given. *(Section 5.3)*

50. $\overline{AE}$ and $\overline{BE}$

51. $\overline{AC}$ and $\overline{DC}$

52. $\overline{AD}$ and $\overline{DC}$

53. $\overline{CE}$ and $\overline{BE}$